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Derivandi generales

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Consent. Ampliss. Ord. Philosoph.,

Præside

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In functionum indole & natura perscrutanda ante omnia functio summæ (vel binomii $x+y$) rite explicanda sese Geometrarum oculis, quasi fontem, unde non tam aquam quam oleum (vel si mavis spiritum merum) haurire liceat, sponte offert. Hæc voro res universim dupliciter tractanda, vel enim functio summæ per functionem cujusvis partis (scil. $f(x+y)$ per fx & fy) exprimenda est, idque finita sub forma, vel etiam per functionem unius partis (fx) seriemque alterius (y) evolvenda. Illa vero quæstio, quamquam plurimi illius casus jam in elementis occurrunt, ut cum functiones Trigonometricæ simili sub forma exprimuntur, (ut, ex. gr. cum indoles tangentis per hanc formulam $T(x+y)=\frac{Tx+Ty}{1-Tx.Ty}$ definitur), attamen ad totam suam vim altioris indaginis invenitur. Hæc vero quæstio jam late in Algebram suas radices agit, circaque Theorema proprie binomiale dictum, quod simplicissimum fere sui casum exhibet, præcipue ibi versatur, succum quasi & nervum inde haustura, indeque cacumen per totius Analyseos Æthera alte exserit. Duo vero præcipua momenta circa hanc ipsam quæstionem observanda veniunt. Cum enim functio quædam binomii cujusdam $f(x+y)$ sub forma $X+yX_1+y^2X_2+y^3X_3+\ldots$ explicanda sit, de numeris coefficientibus quidem his X, X_1, X_2 ... inveniendis proprie agitur, attamen primus X nullum omnino facessit negotium, (facile enim, ponendo $y=o$, evincitur $X=fx$ fore), alter vero X_1 tanto majus, ut sæpe,

quamquam sequentes X_2, X_3 &c. facile inveniantur vel saltem per X_1 exprimantur, tamen X_1 per eundem modum, quo hi omnes inventi sunt, reperire haud liceat, (sicuti jam in potestate $\overline{x+y}^n$ explicanda accidit). Universim igitur coefficiens X_1 suum, reliquique suum sibi vindicant modum; quin imo illius definiendi pro indole ipsius functionis varius sæpe exigitur modus.

Sic ex. gr. proposita functione exponentiali a^{x+y} evolvenda sub forma dicta $X + X_1 y + X_2 y^2 + \ldots$, facile quidem ex ipsius *indole fundamentali, qualis pro diversa cujusvis functionis natura hujusmodi in re semper adhibenda venit*, $a^{x+y} = a^x a^y$, huic propria, apparet, non modo X ipsum a^x æquare, sed etiam reliquos numeros X_1, X_2 &c. hunc eundem factorem a^x admissuros, & quidem ita ut constans sit alter factor, qui vero pro X_1 & pro reliquis diversimode erit definiendus. Facto enim $x = o$, erit omnino $a^x = a^0 = 1$, ideoque X tum $= 1$, reliqui vero coefficientes, cum universim nihil nisi a & x continere possint, (quamdoquidem y expresse, vel explicite ut dicunt, positum intelligitur), jamque x evanuit, nihil nisi a continere poterunt, ideoque ipsorum, ex positione $x = o$ oriundi, valores jam per A_1, A_2, A_3 &c. insignire licet, quæque ita necessario ipsius a sint functiones, ut jam loco æquationis admissæ $a^{x+y} = X + X_1 y + X_2 y^2 + \ldots$, hanc adepti fuerimus $(a^{0+y} =) a^y = 1 + A_1 y + A_2 y^2 + A_3 y^3 + \ldots$, quæ evolutionem exponentialis a^y indicat. Quam vero si in a^x duxeris, habebis

$$a^x + A_1 a^x y + A_2 a^x y^2 + A_3 a^x y^3 + \ldots = a^x a^y = a^{x+y} = X + X_1 y + X_2 y^2 + \ldots,$$

indeque ex identitate partium, (quia y arbitrio relinquitur), erit $X_1 = A_1 a^x$, $X_2 = A_2 a^x$, $X_3 = A_3 a^x$ &c., universimque $X_n = A_n a^x$, sicuti diximus. Quadantenus igitur innotuere hi coefficientes, scil. quatenus cum x variant, haud tamen integre, quamdiu haud noveris, quomodo cum a varient numeri A_1, A_2, A_3 &c., seu quales ipsius a hi sint functiones.

Quem in finem eandem proprietatem fundamentalem ampliatam, ut $a^{x+y+z} = a^x a^{y+z} = a^{x+y} a^z$ sit, denuo adhibeas, & quidem hunc in modum:

Quia jam obtinuimus $a^{x+y} = a^x + A_1 a^x y + A_2 a^x y^2 + ..$, idque pro valoribus quibusvis ipsorum x & y, erit similiter $a^{(x+y)+z} = a^{x+y} + A_1 a^{x+y}.z + A_2 a^{x+y}.z^2 + ..$ (scriptis sc. $x+y$ pro x & z pro y), atque (si $y+z$ pro y adhibetur) $a^{x+(y+z)} = a^x + A_1 a^x \widehat{y+z} + A_2 a^x \widehat{y+z}^2 + ..$; quæ utraque igitur necessario conveniant. Ulterius igitur utrimque evolvendo erit

$$\begin{aligned} & a^x + A_1 a^x y + A_2 a^x y^2 + A_3 a^x y^3 + .. \\ & + A_1 z.(a^x + A_1 a^x y + A_2 a^x y^2 + \ldots \quad) \\ & + A_2 z^2 (a^x + \ldots \\ & \quad + \; + \; .. \\ & \quad = a^x + A_1 a^x y + A_2 a^x y^2 + A_3 a^x y^3 + \; .. \\ & \qquad + z.(A_1 a^x + 2A_2 a^x y + 3A_3 a^x y^2 + \; .. \quad) \\ & \qquad + z^2 (A_2 a^x + \ldots \quad) \end{aligned}$$

Ubi ab utraque parte primæ lineæ conveniunt; sed & secundæ seu coefficientes ipsius z necessario congruant, ideoque sit $A_1(a^x + A_1 a^x y + A_2 a^x y^2 +) = A_1 a^x + 2A_2 a^x y + 3A_3 a^x y^2 + ... + nA_n a^x y^{n-1} +$, ulteriusque ex identitate secundum y, factore communi a^x sejuncto, $A_1 = A_1$, $A_1^2 = 2A_2$, $A_1 A_2 = 3A_3$, $A_1 A_3 = 4A_4, \ldots A_1 A_{n-1} = nA_n$. Unde facile $A_2, A_3, \ldots A_n$ gradatim exprimas, ut jam sit $a^{x+y} = a^x + A_1 a^x y + \frac{1}{1.2}.A_1^2 a^x y^2 + \frac{1}{1.2.3} A_1^3 a^x y^3 + ..$, ideoque, ut innuimus, facile omnes numeri per primum inventi sunt. Qui vero adhuc ignotus restat: Licet quidem $x = o$ & $y = 1$ facere, ut jam igitur $a = a^1 =$

$$1 + A_1 + \frac{A_1^2}{1.2} + \frac{A_1^3}{1.2.3} + \ldots \frac{A_1^n}{1.2.3...n} +,$$ ideoque A_1 radix hujus æquationis sit; qui vero modus jam et magis specialis et aliqua difficultate laborare videtur, cum hanc infiniti ordinis aquationem solvendi ars in promptu haud sit, quare alius jam quærendus est. Ut scilicet & radix hujus satis ardui æquationis & indoles numeri A_1 inveniatur, quatenus sc. ab a pendet, ideoque aliqua ipsius a est functio, quam per La indicare licet, talis ipsius exponentialis

formula est adhibenda, quæ hoc argumentum quam generalissime variari faciat, quæ quidem hæc est: $(ab)^x=a^x.b^x$. Quia igitur $A_1=La$, & $a^x=1+A_1x+\frac{1}{1.2}.A_1^2x^2+$, erit $a^x=1+xLa+$, similiterque $b^x=1+xLb+$ atque $(ab)^x=1+xL(ab)+$, ideoque $1+xL(ab)+\frac{1}{1.2}x^2\overline{L(ab)}^2+\cdots=$ $=(1+xLa+).(1+xLb+)$, unde, evolutione & comparatione rite instituta, facile colliges $L(ab)=La+Lb$, quæ igitur hujus functionis L formula est fundamentalis, ex qua facile concludes, ipsam nihil nisi *logarithmum* esse aliquem, basi omnimodo determinata, ideoque *naturali* dicta, gaudentem, quandoquidem jam alius constans, cum quo variare possit, nullus adsit.

Quod vero hoc in exemplo exhibuimus, idem in unoquoque Problemate functionem quandam f binomii $x+y$ arbitrarii, (seu $f(x+y)$, ut breviter scribitur), evolvendi occurrit, ut scilicet partim prævia aliqua, ex speciali vel caracteristica propositæ functionis f indole petita, evolutione inveniantur duo priores termini ($X+yX_1$ seu) $fx+y.f_1x$, ubi f_1x aliqua est alia functio, quæ hac ipsa ex re cum data X vel fx arcte cohæret indeque speciali suo modo derivatur, qvam & ideo hujus *derivatam* appellant Geometræ & per $X_1, f'x, f_1x$ vel ∂fx $\left(\text{vel etiam } \frac{dfx}{dx}\right)$ indicant, sicuti vice versa inventa vel data $f_1x=\varphi x$, fx hujus φx *primaria* dicitur, & ideo per ${}^1\varphi x$ vel $\partial^{-1}\varphi x$ vel $\int\varphi x$ significatur, quatenus scil. ipsa aliqua *rederivatione* ex data φx reperitur. Partim vero reliqui deinde termini ($X_2y^2+X_3y^3+\ldots$ seu) $f_2x.y^2+f_3x.y^3+\ldots$ determinandi sunt, hocque et universalius & facilius perficitur, idque eodem fere modo, quem jam in exemplo allato adhibuimus hacque aggregati $(x+y)+z=x+(y+z)$ proprietate superstruximus. Huic enim functionem quamcunque f applicando erit $f(\widehat{x+y}+z)=f(x+\widehat{y+z})$, quæ ex assumta $f\widehat{x+u}=fx+uf_1x+u^2f_2x+u^3f_3x+$ evolutione, utrimque adhibita, primum in hanc migrat æquationem:

$$f\overline{x+y}+z.f_1\overline{x+y}+z^2f_2\overline{x+y}+\ldots = fx+\overline{y+z}f_1x+\overline{y+z}^2.f_2x+\ldots$$

deindeque in hanc:

$$(fx+y.f_1x+y^2f_2x+\ldots)+zf_1\overline{x+y}+z^2.f_2\overline{x+y}+\ldots$$
$$=fx+yf_1x+y^2f_2x+\ldots+z.(f_1x+2yf_2x+3y^2f_3x+\ldots)+\ldots$$

unde mox identice comparando colligis

$$f_1\overline{x+y}=f_1x+2yf_2x+3y^2f_3x+\ldots+ny^{n-1}f_nx+\ldots$$

Quæ formula breviter docet, *omnem functionis $f(x+y)$ evolutionem ad similem ipsius $f_1\overline{x+y}$ revocari*, ideoque omnem hujusmodi quæstionis nodum, siquidem evolutio ad potentias integras admittitur, in prima cujusvis functionis derivata rite invenienda contineri. Formula enim hæc docet, si $f\overline{x+y}=fx+yf_1x+y^2f_2x+\ldots$ ponitur, similiterque $\varphi\overline{x+y}=\varphi x+y\varphi_1x+y^2\varphi_2x+\ldots$, fueritque jam inventum $f_1x=\varphi x$, fore $2f_2x=\varphi_1x$, $3f_3x=\varphi_2x$, universimque $nf_nx=\varphi_{n-1}x$, unde $f_nx=\frac{1}{n}.\varphi_{n-1}x$, seu $f_{n+1}x=\frac{1}{n+1}.\varphi_nx$ fore, seu *jam f_1x ex indole speciali ipsius fx* (seu quasi signum functionis f_1 vel φ ex illo f) *invento, ad reliquos coefficientes f_2x, $f_3x\ldots f_nx$, $f_{n+1}x$ inveniendos id tantummodo exigi, ut $f_1\overline{x+y}$*, (quæ plerumque simplicior est, vel etiam ejusdem fere indolis, ac $f\overline{x+y}$, attamen aliter expressa, ut comparatio aliqua habeatur), *complete evolvatur, hujusque evolutionis coefficientes ordine per* (1,) 2, 3, . . . n, $n+1$ *dividantur.*

Sic si vel ex Euclide tantummodo noveris $\overline{x+y}^2=x^2+2xy+y^2$, aliterque tibi persvaseris, derivatam ipsius x^3 esse $3x^2$, seu duos priores ipsius $\overline{x+y}^3$ terminos esse x^3+3x^2y, reliquos $x_2^3.y^2+x_3^3.y^3$ ex duobus ultimis $3(+2xy+y^2)$ ipsius $3(x+y)^2$ ita petas, ut sit $x_2^3=\frac{3.2x}{2}=3x$, atque $x_3^3=\frac{3.1}{3}=1$, seu $x_2^3.y^2=\frac{3y.2xy}{2}=3xy^2$ atque $x_3^3y=\frac{3y.y^2}{3}=y^3$, ideoque totum $\overline{x+y}^3=x^3+3x^2y+3xy^2+y^3$. Similiter vero, si aliunde noveris, *deri-*

vatam x_1^n, seu $\partial(x^n)$ ipsius x^n esse nx^{n-1} pro quibusvis ipsius n valoribus, ut sit $\overline{x+y}^n = x^n + nx^{n-1}y + x_2^n.y^2 + x_3^n.y^3 + \ldots + x_r^n.y^r + \ldots$, erit quidem eadem de causa $(x+y)^{n-1} = x^{n-1} + \overline{n-1}.x^{n-2}y + x^{n-1}_2 y^2 +$ atque $n(x+y)^{n-1} = nx^{n-1} + n.\overline{n-1}.x^{n-2}y + nx^{n-1}_2 y^2 + \ldots$, quod ex nostro Theoremate $= x_1^n + 2x_2^n.y + 3x_3^n.y^2 + \ldots$ necessario æquet, ideoque ex identitate evincitur, $2x_2^n = n\overline{n-1}x^{n-1}$, $3x_3^n = nx^{n-1}_2$ fore, ideoque $x_2^n = \frac{n.(n-1)}{2}.x^{n-1}$, & $x_3^n = \frac{nx_2^{n-1}}{3} = \frac{n}{3}.\left(\frac{\overline{n-1}.\overline{n-2}}{2}.x^{n-2}\right) = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}.x^{n-2}$ (si $n-1$ pro n, in præcedenti formula scribis), similiterque gradatim reliquos invenies coefficientes, ut *theorema Newtoni* seu *binomiale* dictum

$$\overline{x+y}^n = x^n + nx^{n-1}y + \frac{n}{1}\cdot\frac{n-1}{2}x^{n-2}y^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}.x^{n-3}.y^3 + \ldots \text{*)}$$

tibi ita facile constet, idque ex regula nostra generali, ut huic nihil speciale insit, quam quod ad *derivatam* nx^{n-1} (vel x_1^n) *primam* pertinet. (Reliquos enim coefficientes f_2x, f_3x &c. *derivatas altiores* appellarunt geometræ). Similiter posito a^x pro fx & a^xLa pro φx, quod f_1x seu ∂fx valebat, obtinebis valorem ipsius a^{x+y} explicitum eundem, quem modo elementarie demonstravimus. Et similiter alias functiones, ut Trigonometricas &c. evolves.

Quia vero universim invenimus $f_2x = \frac{1}{2}\varphi_1 x$, $f_3x = \frac{1}{3}.\varphi_2 x$, $f_{n+1}x = \frac{1}{n+1}.\varphi_n x$, si $\varphi x = f_1x = \partial fx$,**) erit quidem, scribendo f_1 vel ∂f pro φ_1 atque $\partial\varphi$ pro φ_1,

*) Quod & ita breviter indicare licet: $\overline{x+y}^n = x^n + nx^{n-1}y + n_2 x^{n-2}y^2 + n_3 x^{n-3}y^3 + \ldots + n_r x^{n-r}y^r +$ vel adhuc brevius per $x^n + x_1^n y + x_2^n.y^2 + \ldots + x_r^n.y^r + \ldots$, signis ex universali f_rx petitis.

**) Olim Præs. ϑ vel ∂ pro ∂ & ∂ pro $\underset{c}{\partial}$, penuria typorum coactus, adhibuit.

$f_2x=\frac{1}{2}f_{1,1}x=\frac{1}{2}\partial f_1x$, seu $f_2x=\frac{1}{2}\partial\partial fx$, quod in $\frac{1}{2}\partial^2 fx$ vel $\underset{c}{\partial^2} fx$ contrahunt geometræ, similiterque (scripto $\underset{c}{\partial^2}\varphi$ pro φ_2) $f_3x=\frac{1}{3}.\underset{c}{\partial^2}\varphi x=\frac{1}{3}\underset{c}{\partial^2}\partial fx$ vel $=\frac{1}{3}.\frac{\partial^2}{2}\partial fx$, quod rursum in $\underset{c}{\partial^3}fx$ vel $\frac{\partial^3 fx}{1.2.3}$ contrahitur, similiterque ulterius, ut universim sit $f_nx=\underset{c}{\partial^n}fx$ & $\varphi_nx=\underset{c}{\partial^n}\varphi x=\underset{c}{\partial^n}\partial fx$, ideoque $\underset{c}{\partial^{n+1}}fx=f_{n+1}x=\frac{1}{n+1}.\varphi_nx=\frac{1}{n+1}\underset{c}{\partial^n}\varphi x=\frac{1}{n+1}.\underset{c}{\partial^n}\partial fx$.

Jam igitur loco formulæ admissæ $f(x+y)=fx+yf_1x+y^2f_2x+y^3f_3x+..$ hanc maximi momenti nacti fuimus, $f(x+y)=fx+y\partial fx+y^2\underset{c}{\partial^2}fx+y^3\underset{c}{\partial^3}fx+..+y^n\underset{c}{\partial^n}fx+..$, ubi $\underset{c}{\partial^n}$ signi hæc est vis, ut sit $\underset{c}{\partial^{n+1}}f=\frac{1}{n+1}\underset{c}{\partial^n}\partial f$, id quod celebratissimum *Theorema Taylorii Binomiale* constituit.

Sed & terminos in evolutione nostra fundamentali (ipsius $f(x+y+z)$ duplici) aliter colligendo facile evincitur, & $\underset{c}{\partial^n}\partial fx=\partial\underset{c}{\partial^n}fx$ fore. Si enim in sinistra parte terminum $z^nf_n(x+y)$ ex eadem formula $\varphi(x+y)=\varphi x+y\partial\varphi x+y^2\varphi_2x+y^3\varphi_3x+$: . . ulterius evolvis, ejus loco obtinebis $z^n(f_nx+y\partial f_nx+y^2\varphi_2x+)$ si f_n pro φ scriptum fuerit, indeque hunc ipsius y coefficientem hauries $z^n\partial f_nx$, ideoque $z\partial f_1x$ ex $zf_1\overline{x+y}$, $z^2\partial f_2x$ ex $z^2f_2\overline{x+y}$ &c., quos omnes igitur si in unum colligis, inde obtinebis

$f_1x+z\partial f_1x+z^2\partial f_2x+\ldots+z^n\partial f_nx+\ldots$, id quod necessario coefficienti ipsius y ab dextra parte, qui (ex $\overline{z+y}^n=z^n+nz^{n-1}y+$ . .) facile invenitur $=f_1x+z.2f_2x+z^2.3f_3x+\ldots+z^{n-1}nf_nx+z^n(n+1)f_{n+1}x+..$, æquivaleat, ideoque ulterius ipsius z numeros comparando erit $2f_2x=\partial f_1x$, $3f_3x=\partial f_2x$, &c. universimque $(n+1)f_{n+1}x=\partial f_nx$, seu $f_{n+1}x=\underset{c}{\partial^{n+1}}fx=\frac{1}{n+1}\partial\underset{c}{\partial^n}fx$, ut con-

tendimus. *Unaquæque igitur functio derivata altior est pars aliquota derivatæ primæ mox præcedentis,* ut $n_{r+1}x^{n-r-1} = \frac{1}{r+1} . \partial n_r x^{n-r}$ in theoremate Newtoniano.

Quoniam vero hæc res in elementis passim satis dilucide proponitur, ulteriore explicatione supersedeamus, hoc tantummodo monentes, si similiter coefficientem ipsius $y^r\ z^n$ utrimque quæsiveris, inveniri $n_r : f_n x = \underset{c}{\partial}^r f_{n-r} x$, seu $n_r \underset{c}{\partial}^n fx = \underset{c}{\partial}^r \underset{c}{\partial}^{n-r} fx$, exsistente n_r coefficiente binomiali $n . \frac{n-1}{2} . \frac{n-2}{3} . \quad \frac{n-r+1}{r}$. Cui hoc tantum subjungimus, $\underset{c}{\partial}^n fx$ in $\frac{\partial\partial\partial}{1.2.3} \cdots \frac{\partial}{n} fx$ seu $\frac{\partial\partial\partial\,..\,\partial fx}{1.2.3 \ldots n}$, cujus loco $\frac{\partial^n fx}{1.2.3\,..\,n}$ vulgo scribunt, sponte dissolvi. Id quod facile ope hujus *Theorematis* evincitur: Si fuerit $\varphi x = a.fx$ & *a constans* (seu cum x haud varians, præterea vero quantitas quæcunque, ut numerus integer vel fractus vel surdus), erit $\partial \varphi x = a.\partial fx$, atque $\underset{c}{\partial}^n \varphi x = a.\underset{c}{\partial}^n fx$.

Tum enim erit $\varphi(x+y) = a.f(x+y)$, ideoque $\varphi x + y\partial\varphi x + y^2\partial^2\varphi x + .. = a.(fx + y\partial fx + y^2\underset{c}{\partial}^2 fx + ..)$, unde res patet.

Simili vero substitutione adhibita, has regulas derivandi facile elicies

$$\partial\,(fx + a) = \partial fx.$$
$$\partial\,(fx \pm \varphi x) = \partial fx \pm \partial\varphi x.$$
$$\partial\,(fx.\varphi x) = fx.\partial\varphi x + \varphi x.\partial fx.$$
$$\partial\left(\frac{\varphi x}{fx}\right) = \frac{fx\partial\varphi x - \varphi x.\partial fx}{fx^2}$$

$\left(\text{seu} \quad \partial\left(\frac{1}{fx}\right) = -\frac{\partial fx}{(fx)^2}\right)$, quas ideo obitur tantum commemoramus.